# On upward straight-line embeddings of oriented paths

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## Abstract

We investigate upward straight-line embeddings (UPSEs) of oriented paths. Along the lines of similar results in the literature, we find a condition —related to the number of vertices in between sources and sinks of an oriented path— that guarantees that an oriented path satisfying the condition on $n$ vertices admits an UPSE into any $n$-point set in general position. We also show that the following holds for every $\varepsilon > 0$. If $S$ is a set of $n$ points chosen uniformly at random in the unit square, and $P$ is an oriented path on at most $(1/3 - \varepsilon)n$ vertices, then with high probability $P$ has an UPSE into $S$.

## Introduction

A *straight-line embedding* of a graph $G$ into a point set $S$ in the plane is an embedding of $G$ in which each vertex is mapped to a point in $S$, each edge is mapped to the straight-line segment between its endpoints, and no two edges cross each other.

The question of deciding whether a given graph admits a straight-line embedding into a given point set $S$ is an important problem in computational geometry [5, 6, 7, 8, 9, 10, 11]. In one variant of this problem, $G$ is a directed graph (or *digraph*, for short), and the question is if $G$ has an *upward* straight-line embedding on $S$, that is, a straight-line embedding of $G$ into $S$ such that, for each (directed) edge $\overrightarrow{uv}$ of $G$, the $y$-coordinate of $u$ is smaller than the $y$-coordinate of $v$. Following [12], for brevity we will refer to an upward straight-line embedding simply as an UPSE.

Binucci et al. [4] presented several interesting statements on which digraphs admit an UPSE on a given point set. Among other results, they proved that even if the point set $S$ is in convex position, then there exist digraphs on $|S|$ vertices whose underlying undirected graphs are trees, and do not have an UPSE into $S$. On the other hand, they proved that if the underlying undirected graph is a path on $|S|$ vertices (and $S$ is in convex position), then an UPSE into $S$ always exists. This last result was refined by Angelini et al. in [3], where (among other results) this was extended to the case in which the underlying undirected graph is a caterpillar.

In this work we focus on the case in which $G$ is an oriented path, that is, the underlying unoriented graph of $G$ is a path. A *switch* in an oriented path is a vertex that is either a source or a sink. Note that the first and last vertices are always switches.

It is easy to show that if an oriented path on $n$ vertices has at most three switches, then it admits an UPSE into every $n$-point set in general position. Along these lines, in [3] some results are given about oriented paths with a small number of switches. For instance, it is proved that if an oriented path $P$ on $n$ vertices has five switches, and at least two of the monotone paths composing $P$ are single edges, then $P$ admits an UPSE into every $n$-point set in general position.

## 1 Our results

We present a condition, also related to switches, that guarantees that an oriented path on $n$ vertices admits an UPSE into every $n$-point set in general position. Let $P$ be an oriented path, and let $P_1, P_2, \ldots, P_r$ be the decomposition of $P$ into maximal monotone paths. That is, for $i = 1, 2, \ldots, r$, $P_i$ is an oriented path none of whose internal vertices is a switch, and is maximal

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with respect to this property, and $P$ is the concatenation $P_1P_2 \dots P_r$. We call $P_1, P_2, \dots, P_r$ the *canonical* decomposition of $P$. If $P$ is a path, then we let $|P|$ denote the number of edges of $P$.

**Theorem 1** *Let $P$ be an oriented path with $n$ vertices, and let $P_1, P_2, \dots, P_r$ be the canonical decomposition of $P$. Let $S$ be any $n$-point set in general position. If $|P_i| \geq \sum_{j>i} |P_j|$ for every $i = 1, 2, \dots, r-1$, then $P$ admits an UPSE into $S$.*

Another result given in [3] is that every oriented path on $n$ vertices with at most $k$ switches admits an UPSE into every point set in general position with $n2^{k-2}$ points. This was later improved (for $k > \Omega(\log n)$) by Mchedlidze in [12], who proved that if $P$ is an oriented path on $n$ vertices, and $S$ is any set of $(n-1)^2 + 1$ points in general position, then $P$ admits an UPSE into $S$.

Our next statement is along the lines of this last result by Mchedlidze, in the sense that we consider the question of whether a given oriented path admits an UPSE into a point set in general position, whose cardinality may be greater than the number of vertices in the path. We show that the quadratic bound in [12] can be improved (with high probability) to a linear bound, if $S$ is a random point set in the unit square.

**Theorem 2** *Let $\varepsilon > 0$, and let $n$ be an integer. Let $P$ be an oriented path on at most $(1/3 - \varepsilon)n$ vertices. If $S$ is a random set of $n$ points in the unit square, then w.h.p. (with high probability) $P$ has an UPSE into $S$.*

In order to prove Theorems 1 and 2 we now introduce the concept of a signature, which encodes an oriented path into a binary sequence.

## 2 Signatures

For the proofs of Theorems 1 and 2 it is convenient to record the orientations of the edges in an oriented path in a binary sequence. A *signature* is a sequence $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ in $\{+, -\}^k$, for some positive integer $k$. The integer $k$ is the *size* of $\sigma$.

Now let $P$ be an oriented path, and let $p_1, p_2, \dots, p_n$ be the vertices of $P$ in the order in which they appear in the underlying oriented path of $P$. We define $\sigma(P)$ as the signature obtained from $P$ as follows. If for $i \in \{1, \dots, n-1\}$ the edge in $P$ joining $p_i$ and $p_{i+1}$ is $\overrightarrow{p_ip_{i+1}}$ (respectively, $\overrightarrow{p_{i+1}p_i}$), then the $i$th entry of $\sigma(P)$ is $+$ (respectively, $-$).

Evidently, any signature $\sigma$ is the signature $\sigma(P)$ of some oriented path $P$.

Now let $S$ be a point set in general position, and let $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ be a signature. We say that $\sigma$ is *realizable* on $S$ if there exist a geometric (that is, straight-line, noncrossing) path $Q = (q_1, q_2, \dots, q_{k+1})$, whose vertices are points in $S$, and for each $i \in \{1, \dots, k\}$, $\sigma_i = +$ (respectively, $\sigma_i = -$) if and only if the $y$-coordinate of $q_i$ is smaller (respectively, greater) than the $y$-coordinate of $q_{i+1}$.

The following is an immediate consequence of these definitions.

**Observation 3** *Let $P$ be an oriented path, and let $S$ be a point set in general position. Then $P$ admits an UPSE on $S$ if and only if $\sigma(P)$ is realizable on $S$.*

This observation allows us to write Theorems 1 and 2 in terms of signatures. For Theorem 1, we need a corresponding notion, for signatures, of the canonical decomposition of an oriented path.

Given a signature $\sigma$, define $\tau_i(\sigma)$ as the $i$th run of either $+$'s or $-$'s, so that $\sigma = \tau_1\tau_2 \dots \tau_r$, where $r$ is the number of runs. We say that $\tau_1\tau_2 \dots \tau_r$ is the *canonical decomposition* of $\sigma$. For example, for $\sigma = (+++--+)$, $\tau_1(\sigma) = (+++)$, $\tau_2(\sigma) = (--)$ and $\tau_3(\sigma) = (+)$.

We now state Theorems 1 and 2 in terms of signatures. The fact that Theorems 4 and 5 are equivalent to Theorems 1 and 2 follows immediately from Observation 3.

**Theorem 4 (Implies Theorem 1)** *Let $\tau$ be a signature of size $n-1$, for some integer $n \geq 2$, and let $\tau_1\tau_2 \dots \tau_r$ be its canonical decomposition. Let $S$ be any $n$-point set in general position. If $|\tau_i| \geq \sum_{j>i} |\tau_j|$ for every $i \in \{1, \dots, r-1\}$, then $\tau$ is realizable on $S$.*

**Theorem 5 (Implies Theorem 2)** *Let $\varepsilon > 0$, and let $n$ be an integer. Let $\tau$ be any signature of size at most $(1/3 - \varepsilon)n$. If $S$ is a random set of $n$ points in the unit square, then w.h.p. $\tau$ is realizable on $S$.*

## 3 Proof of Theorem 4

We proceed by induction on $r$.

We restrict ourselves to paths $P = (p_1, \dots, p_n)$ such that $p_1$ is in the boundary of the convex hull and no edge $\overline{p_ip_{i+1}}$ (the straight segment joining $p_i$ and $p_{i+1}$) intersects the interior of the convex hull of $\{p_j\}_{j>i}$. This guarantees that $P$ does not self-intersect.

**Lemma 6** *Let $Q$ be a set of points. The signature $\sigma = \tau_1\tau_2$ with $|\tau_1| > |\tau_2|$ (with $\tau_1$ consisting of $+$'s) can be realized by a path that starts in the lowest point of $Q$.*

**Proof.** Let $q$ be the lowest point of $Q$, and let $s(q, Q)$ be the length of the shortest path from $q$ to the highest point in $Q$ using only vertices and edges from the boundary of the convex hull of $Q$.

Clearly $s(q, Q) \leq |Q|/2$. If $s(q, Q) = |\tau_1| + 1$, we can just ascend using the path given by $s(q, Q)$ and

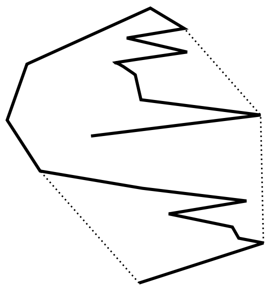

Figure 1: Visualization of a path from Lemma 6.

descend using the remaining points. If not, necessarily $s(q, Q) \leq |\tau_1|$. Then, we define $q'$ as the lowest point of $Q' = Q \setminus \{q\}$. There are two cases. If $s(q', Q') \leq |\tau_1|$, continue the path to $q'$ and proceed inductively. Otherwise, let $U$ be the set of points in $\Delta(Q') \setminus \Delta(Q)$, where $\Delta(X) \subset X$ denotes the set of points in the boundary of the convex hull of $X$ (see Figure 2).

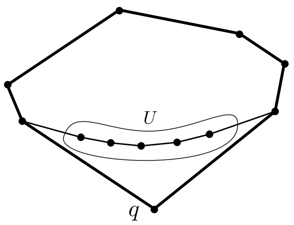


Figure 2: Every point of $U$ can be seen from $q$.

It is easy to see that no segment $\overline{qu}$ for $u \in U$ intersects the interior of $\Delta(Q')$. Choose $u \in U$ for which $s(u, Q') = |\tau_1|$ and just follow the boundary until the highest point is reached.

Once the highest point is reached using $|\tau_1|$ ascending segments, we can finish the path by descending through the remaining points. □

Consider $\sigma'$ constructed by flipping the signs of $\tau_r$ in $\sigma$. Suppose, for example, that $\tau_r$ consists of −'s and we flip them to +'s. Clearly, $\sigma'$ satisfies the conditions in Theorem 4. By the induction hypothesis, $\sigma'$ can be realized in $Q$ by a path $P'$ satisfying the above restriction. The last run of $P'$ consists only of ascending edges. Then let $V$ be the set of points involved in this run, and apply Lemma 6.

## 4 An algorithm to realize a prefix of a signature

We have devised an algorithm that, given a point set $S$ in the unit square, and a signature $\sigma$, yields a path $P$ that realizes a prefix of $\sigma$. This algorithm was designed having in mind the case in which $S$ is a random point set; for an arbitrary $S$, it can give extremely poor results. If at any point when running the algorithm we cannot continue, we stop and return the current path $P$.

Sort the points of $S$ by their $x$-coordinate. The algorithm processes one point at a time in this order. In each step we decide whether or not the current point will ultimately belong to path $P$.

Suppose, without loss of generality, that $\sigma$ starts with $+$. Divide the unit square into horizontal thirds and find the first point $q_f$ in the bottom third (whose $y$ coordinate is in $[0, 1/3)$). This will be the first point of the path $P$.

Now we attempt to extend $P$. Let $U$ be the set of the first $|\tau_1(\sigma)| - 1$ points which come after $q_f$ and are in the middle third (whose $y$ coordinates are in $[1/3, 2/3)$). Then find the next point $q_\ell$ which is in the top third. Set $P$ to be $(q_f, U, q_\ell)$, where the points of $U$ are taken in ascending order with respect to their $y$ coordinates.

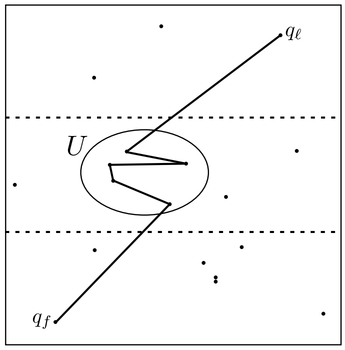


Figure 3: An example with $|\tau_1| = 6$

After this process we are in a situation where the last point of $P$ is in the top third, and the next part of the signature ($\tau_2$) consists of −'s, which is an analogous situation to the one we started with and so we can repeat the procedure. Note that this path cannot self-intersect.

As we mentioned above, this algorithm may work extremely poorly for some point sets and signatures. For example, if all the points in the point set $S$ have $y$-coordinates less than $1/3$, and $\sigma$ starts with $+$, then the resulting path will not have a single edge. But this algorithm is designed for dealing with random point sets.

## 5 Realizing signatures in random point sets: proof of Theorem 5

In order to analyze the performance of the given algorithm for random point sets, it is convenient to think that we first randomly select the $x$-coordinates, then order these from left to right, and then randomly select the $y$-coordinates. This last step (of selecting the $y$-coordinates) may be thought of as *unveiling* the points. The key observation is that *each time we unveil a point, with probability* $1/3$ *this point will end up being part of the final path.* Indeed, at each step, when we are about to unveil a point, there is an interval $I \in \{[0, 1/3), [1/3, 2/3), [2/3, 1]\}$ such that the point to be unveiled will be part of the final path if and only if the $y$-coordinate of this point is in $I$.

The main ingredient in the proof of Theorem 5 is the following lemma, which estimates the expected size of the non-crossing path obtained from the algorithm described in the previous section, for the case in which $S$ is randomly generated.

**Lemma 7** *Let $S_n$ be a random set of $n$ points in the unit square, and let $\sigma$ be any signature of length $n-1$. Let $\text{PREFIX}(S_n, \sigma)$ be the random variable that measures the size of the largest prefix of $\sigma$ that can be realized in $S_n$. Then, for every $\varepsilon > 0$, w.h.p. (with high probability)*

$$\text{PREFIX}(S_n, \sigma) \geq (1/3 - \varepsilon)n.$$

**Proof.** We run the algorithm described in the previous section, on $S_n$ and $\sigma$. Let $P$ be the path obtained at the end of the algorithm. Thus $P$ realizes a prefix of $\sigma$. We recall that for each point $i$ of $S_n$, the probability that $i$ is in $P$ is $1/3$. A standard argument using Chernoff's bound for the sum of independent random variables (see Theorem A.1.11 in [2]) shows that

$$\Pr\left[|P| < \big(1/3 - \varepsilon\big)n\right] < e^{(-3\varepsilon^2/4)(1-3\varepsilon/2)n},$$

from which the lemma immediately follows. □

**Proof.** [Proof of Theorem 5] Let $S_n$ be a random set of $n$ points in the unit square, and let $\tau$ be a signature of length $\ell \leq (1/3 - \varepsilon)n$. Let $\sigma$ be the signature of length $n-1$ obtained by appending $n-1-\ell$ +'s to $\tau$. By Lemma 7, w.h.p. the largest prefix of $\sigma$ that can be realized in $S_n$ has size at least $(1/3 - \varepsilon)n$. Thus w.h.p. $\tau$ can be realized in $S_n$. □

## 6 Concluding remarks and open questions

An important open question is whether or not, for every ordered path $P$ with $n$ vertices and every $n$-point set $S$ in general position, $P$ admits an UPSE into $S$. In [1], it is reported that this has an affirmative answer for every $n \leq 10$. In the terminology of signatures, this reads as follows.

**Question 8** *Is it true that for every $n$-point set $S$ in general position, and every signature $\tau$ of size $n-1$, $\tau$ is realizable on $S$?*

The result by Mchedlidze [12] mentioned in Section 1 implies that if $S$ is an $n$-point set in general position, and $\tau$ is a signature of size $n-1$, then *every* subsequence of $\tau$ of size at most (roughly) $\sqrt{n}$ can be realized on $S$. With an eye on Question 8, one could ask for the existence of larger subsequences of $\tau$ that can be realized on $S$.

It is easy to see that if $\tau$ is a signature of size $n-1$, and $S$ is any $n$-point set in general position, then $\tau$ has a subsequence of size at least $(n-1)/2$ that can be realized on $S$. Indeed, it suffices to consider the maximal subsequences that consist of all +'s or all −'s; both subsequences are trivially realizable on $S$, and so it suffices to take the larger one. We have been unable to show the existence of a (substantially) larger subsequence of $\tau$ that can be realized on $S$. In this spirit, we put forward the following weaker, and seemingly more approachable, version of Question 8.

**Conjecture 9** *There exists a constant $c > 1/2$ with the following property. Let $S$ be an $n$-point set in general position, and let $\tau$ be a signature of size $n-1$. Then there exists a subsequence of $\tau$ of size at least $cn$ that can be realized on $S$.*

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